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THE ASYMPTOTIC FORM OF THE STRESS–STRAIN STATE NEAR A SPATIAL SINGULARITY OF THE BOUNDARY OF THE "BEAK TIP" TYPE†

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The asymptotic form of the stress–strain state of a three-dimensional elastic body in the neighbourhood of a singular point of a special type is investigated. Near such a point the boundary of the body consists of four surfaces, namely, two planes forming a dihedral angle and two smooth surfaces tangent to one another at a point on the edge of the angle. (Similar singularities are present on the cutting edges of some devices.) A procedure for determining the structure of asymptotic solutions is presented.

1. FORMULATION OF THE PROBLEM

LET m BE a natural number, let $h_\pm$ be smooth functions of the angle variable $\varphi \in [-\alpha, \alpha]$, where $\alpha \in (0, \pi]$, and let $h(\varphi) = h_+(\varphi) + h_-(\varphi) > 0$ for $|\varphi| < \alpha$. Moreover, let Ω be a bounded body in $\mathbf{R}^3$ defined by the relations

$$-H_-(x_1, x_2) < x_3 < H_+(x_1, x_2)$$

$$H_\pm(x_1, x_2) = r^{2m} h_\pm(\varphi), \quad |\varphi| < \alpha \tag{1.1}$$

in a neighbourhood of the origin O, where r, φ, z are cylindrical coordinates. In this case we say that O is a singular point of the "beak tip" type (see Fig. 1).

The material of the elastic body Ω will be assumed to be homogeneous and isotropic with Lamé constants λ and μ. We will also assume that the bases $\Gamma_\pm = \{x : z = \pm r^{2m} h_\pm(\varphi),\ r > 0,\ |\varphi| < \alpha\}$ are free of stresses. On the lateral surfaces $\Sigma_\pm = \{x : -r^{2m} h_-(\varphi) < z < r^{2m} h_+(\varphi),\ r > 0,\ \varphi = \pm\alpha\}$ we specify one of the following two conditions: either the displacement vector u is equal to zero (the Dirichlet condition), or the components $\sigma_{r\varphi}$, $\sigma_{\varphi\varphi}$, $\sigma_{\varphi z}$ of the stress tensor are equal to zero (the Neumann condition). We do not exclude the case when φ in (1.1) varies over the unit circle S_1^1, i.e. the two smooth surfaces forming the boundary of the body are tangent at O and $\Sigma_\pm$ are empty sets.

Next, a formal construction (without rigorous justification) of the asymptotic form of the stress–strain state near O is given. The methods for studying boundary-value problems in narrow domains (see [1–4], etc.) are used. The use of such methods is justified by the fact that the intersection of Ω and a sphere S_δ^2 of small radius δ centred at O is a narrow (of width $O(\delta^{2m-1})$ and length $O(\delta)$) curvilinear zone, the behaviour of the solution near the singularity being determined by the geometry of $\Omega \cap S_\delta^2$ (cf. [5–7]). We remark that the same methods enable one to study the asymptotic form at infinity of solutions of elastic problems in a layer or in a layer sector [8, 9].

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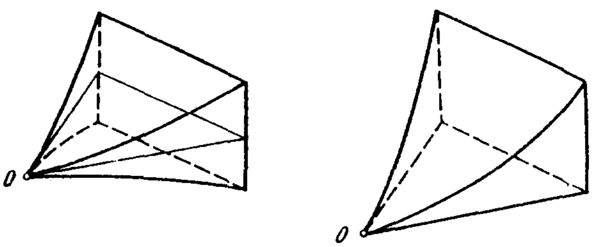

FIG. 1.

The asymptotic form of the displacement field u near the tip of the beak involves (as in problems with other types of singularities of the boundary) a linear combination of special asymptotic solutions $\mathbf{u}$ with coefficients that depend on the data of the problem as a whole (on the loads and the whole domain Ω). However, the solutions $\mathbf{u}$ themselves are defined only by the shape of the beak (formulae (1.1)) and the local properties of the elastic material. In the case of a conic point the asymptotic solutions have the simple form

$$\rho^{\Lambda}\sum_{k=0}^{q}\frac{1}{k!}(\ln\rho)^k\Phi^{(q-k)}(\omega),\quad q=0,\dots,\kappa-1 \tag{1.2}$$

where ρ and ω are spherical coordinates, Λ is a number and $\Phi^{(0)},\dots,\Phi^{(\kappa-1)}$ are vector-valued functions. For the tip of the beak, the structure of the asymptotic solution becomes much more complicated and must be studied separately, which is the purpose of the present paper.

Throughout this paper we will use the method of introducing a small dummy parameter $\varepsilon>0$. It is clear that the relationships (1.1) are invariant under the inhomogeneous coordinate scaling

$$x=(x_1,x_2,x_3)\mapsto(y,\zeta)$$
$$y=(y_1,y_2)=\varepsilon^{-\gamma}(x_1,x_2),\quad \zeta=\varepsilon^{-1}x_3,\quad \gamma=(2m)^{-1} \tag{1.3}$$

Since $\gamma<1$, it is natural to declare ζ to be a "fast" variable, expand the operators of the problem to series in the powers of $\varepsilon^{1-\gamma}$, and then use the algorithms from [1–4]. In this way one can eliminate the variable $z=x_3$, which varies over a narrow range that vanishes as $r\to 0$, and obtain a system of three differential equations for the leading term (υ, w) of the asymptotic solution $\mathbf{u}$ in the angle $K=\{(x_1,\ x_2):r>0,\ |\varphi|<\alpha\}$. The system is derived and transformed to self-adjoint form in Sec. 2. Section 3 is devoted to the case of tangent smooth surfaces, namely, we discuss general questions concerned with finding power solutions of a degenerate system on $K=\mathbf{R}^2\setminus 0$ (the "total" angle), give explicit formulae in the presence of symmetry $h_+=h_-$, and interpret those solutions that increase near O. In Sec. 4 it is assumed that the lateral surface $\Sigma_\pm$ is rigidly supported. Correspondingly, the Dirichlet problem is set on the sides of the angle K, the procedure for finding power solutions of the boundary-value problem in K remaining the same. A new feature in Sec. 4 is the construction of asymptotic corrections appearing in $\mathbf{u}$ due to the boundary layers near the lateral surface $\Sigma_\pm$. As in [1, 2], the condition of exponential decay of the boundary layer makes it possible to derive the boundary conditions on ∂K themselves. In the case of a free lateral surface this is done in Sec. 5.

2. DERIVATION OF A SYSTEM OF LIMITING EQUATIONS

In terms of the coordinates (1.3), the Lamé operator can be written as follows:

$$L(\nabla_x)=\varepsilon^{-2}\{L^0(\partial_\zeta)+\varepsilon^{1-\gamma}L^1(\nabla,\partial_\zeta)+\varepsilon^{2-\gamma}L^2(\nabla)\} \tag{2.1}$$

$$L(\nabla_x)=\mu\nabla_x\cdot\nabla_x+(\lambda+\mu)\nabla_x\nabla_x\cdot$$

$$\nabla_x=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)^{\pm},\quad \nabla=(\partial_1,\partial_2)^t,\quad \partial_i=\partial/\partial y_i,\quad \partial_\zeta=\partial/\partial\zeta$$

$$L^0(\partial_\zeta)=\operatorname{diag}\{\mu,\mu,2\mu+\lambda\}\partial_\zeta^2$$

$$L^1(\nabla,\partial_\zeta)=(\lambda+\mu)\partial_\zeta\begin{Vmatrix}\mathbf{0} & \nabla\\ \nabla^t & 0\end{Vmatrix},\quad L^2(\nabla)=\begin{Vmatrix}\mathbf{L}(\nabla) & \mathbf{0}\\ \mathbf{0} & \mu\nabla\cdot\nabla\end{Vmatrix},$$

Here $\mathbf{L}(\nabla)$ is the two-dimensional Lamé operator with constants λ and μ, t denotes transposition, and the scalar product is denoted by a dot (i.e. $\nabla_x\cdot$ and $\nabla\cdot$ are the divergence operators in $\mathbf{R}^3$ and $\mathbf{R}^2$). Besides, here and henceforth we shall adopt the following convention for writing (3×3)-matrices: the left upper block has dimensions 2×2, while the right lower corner contains a scalar. We denote by $B^{\pm}(\varepsilon, x, \nabla_x)$ the (3×3)-matrix-valued differential operator of the boundary conditions on $\Gamma_\pm$ (in terms of stresses). The outward unit vector normal to $\Gamma_\pm$ has the form

$$n_\pm(\varepsilon,x)=[1+\varepsilon^2|\nabla H_\pm(y)|^2]^{-1/2}(-\varepsilon\nabla H_\pm(y),\pm1)$$

Thus

$$[1+\varepsilon^2|\nabla H_\pm(y)|^2]^{1/2}B^\pm(\varepsilon,x,\nabla_x)=$$
$$=\varepsilon^{-1}\{B^{0\pm}(\partial_\zeta)+\varepsilon^{1-\gamma}B^{1\pm}(y,\nabla,\partial_\zeta)+\varepsilon^{2-2\gamma}B^{2\pm}(y,\nabla)\} \tag{2.2}$$

$$B^{0\pm}(\partial_\zeta)=\pm\operatorname{diag}\{\mu,\mu,2\mu+\lambda\}\partial_\zeta$$

$$B^{1\pm}(y,\nabla,\partial_\zeta)=-\begin{Vmatrix}\mathbf{0} & \lambda\nabla H_\pm\\ \mu(\nabla H_\pm)^t & 0\end{Vmatrix}\partial_\zeta\pm\begin{Vmatrix}\mathbf{0} & \mu\nabla\\ \lambda\nabla^t & 0\end{Vmatrix}$$

$$B^{2\pm}(y,\nabla)=-\begin{Vmatrix}\partial_1H_\pm\mathbf{B}^{(1)}+\partial_2H_\pm\mathbf{B}^{(2)} & \mathbf{0}\\ \mathbf{0} & \mu\nabla H_\pm\cdot\nabla\end{Vmatrix}$$

$$\mathbf{B}^{(1)}(\nabla)=\begin{Vmatrix}(2\mu+\lambda)\partial_1 & \lambda\partial_2\\ \mu\partial_2 & \mu\partial_1\end{Vmatrix},\quad \mathbf{B}^{(2)}(\nabla)=\begin{Vmatrix}\mu\partial_2 & \mu\partial_1\\ \lambda\partial_2 & (2\mu+\lambda)\partial_2\end{Vmatrix}$$

We remark that $\mathbf{B}^{(i)}(\nabla)\upsilon$ are the two-dimensional stress vectors (plane strain).

The operators (2.1) and (2.2) contain the small parameter for some of the leading derivatives. As in [1–4], we will seek a formal series for the solution outside conic neighbourhoods of $\Sigma_\pm$ in the form

$$\mathbf{u}(\varepsilon,y,\zeta)=\sum_{j=0}^{\infty}\varepsilon^{\alpha j}U^j(y,\zeta),\quad \alpha=1-\gamma \tag{2.3}$$

$$U^0(y,\zeta)=(\mathbf{0},w(y)),\quad U^1(y,\zeta)=(\upsilon(y)+V^1(y,\zeta),0)$$

$$U^2(y,\zeta)=(\mathbf{0},W^2(y,\zeta)),\quad U^3(y,\zeta)=(V^3(y,\zeta),0)$$

The function w (the normal displacement) and the vector-valued function $\upsilon=(\upsilon_1,\ \upsilon_2)$ (the displacements in the tangent plane to the beak) are to be determined. The other unknowns in (2.3) can be expressed in terms of υ and w. They can be found as follows: the expansions (2.1)–(2.3) are substituted into the homogeneous Lamé equation and the homogeneous boundary conditions on $\Gamma_\pm$, and then the coefficients corresponding to the same powers of ε are collected. As a result, one obtains a recurrent sequence of ordinary differential equations (in ζ)

in the interval $\Upsilon(y)=[-H_-(y),\ H_+(y)]$ depending on $y=(y_1,\ y_2)$ as on parameters

$$L^0U^j=-L^1U^{j-1}-L^2U^{j-2},\quad \zeta\in\Upsilon(y)$$
$$B^{0\pm}U^j=-B^{1\pm}U^{j-1}-B^{2\pm}U^{j-2},\quad \zeta=\pm H_\pm(y) \tag{2.4}$$

Here $j=0,1,\ldots$ and $U^j=0$ for $j<0$. It is obvious that Eqs (2.4) are satisfied for $j=0$. Solving (2.4) for $j=1,2$, we find that

$$V^1(y,\zeta)=-\zeta\nabla w(y)$$
$$W^2(e,\zeta)=\lambda[\lambda+2\mu]^{-1}(\tfrac{1}{2}\zeta^2\nabla\cdot\nabla w(y)-\zeta\nabla\cdot\upsilon(y)) \tag{2.5}$$

Problem (2.4) with $j=3$ is not always solvable. For the vector V^3 in (2.3) to exist it is necessary that the components of the (vector-valued) right-hand sides F and $G^\pm$ of Eqs (2.4) satisfy the condition

$$-\int_{\Upsilon(y)}F_i(y,\zeta)d\zeta+G_i^+(y)-G_i^-(y)=0 \tag{2.6}$$

for $i=1,2$.

Taking (2.4) and (2.5) into account, we can rewrite (2.6) as a system of two equations relating υ and w

$$-Q(\mathrm{H}_1;y,\nabla)\upsilon(y)+Q(\mathrm{H}_2;y,\nabla)\nabla w(y)=0,\quad y\in K \tag{2.7}$$
$$\mathrm{H}_1=H_++H_-,\quad \mathrm{H}_2=\tfrac{1}{2}(H_+^2-H_-^2)$$
$$Q=(Q_1,Q_2);\quad Q_j(\mathrm{H};y,\nabla)\upsilon(y)=\partial_1\{\mathrm{H}(y)\tau_{j1}(\upsilon;y)\}+\partial_2\{\mathrm{H}(y)\tau_{j2}(\upsilon;y)\}$$
$$\tau_{jk}(\upsilon;y)=\mu(\partial_j\upsilon_k(y)+\partial_k\upsilon_j(y))+\delta_{j,k}2\lambda\mu(\lambda+2\mu)^{-1}\nabla\cdot\upsilon(y),\quad j,k=1,2$$

Note that $\tau=\|\tau_{jk}\|$ is the stress tensor for the plane stress state. Subject to the condition (2.7), the problem under consideration has the solution

$$V^3(y,\zeta)=(\lambda+2\mu)^{-1}[\tfrac{1}{6}\zeta^3(3\lambda+4\mu)\nabla\nabla\cdot\nabla w(y)-$$
$$-\tfrac{1}{2}\zeta^2\{(\lambda+2\mu)\ \nabla\cdot\nabla\upsilon(y)+2(\lambda+\mu)\nabla\nabla\cdot\upsilon(y)\}+$$
$$+\zeta(2\mu)^{-1}[Q(H_+-H_-;y,\nabla)\upsilon(y)-\tfrac{1}{2}Q(H_+^2+H_-^2;y,\nabla)\nabla w(y)] \tag{2.8}$$

We consider the system (2.4) for $j=4$. Now, in general, the (scalar) problem for the third component W_4 of U^4 is unsolvable. By virtue of (2.5) and (2.8), condition (2.6) can be transformed into a new equation connecting υ and w

$$\nabla\cdot\{\mathrm{H}_2(y)Q(\mathrm{H}_1;y,\nabla)\upsilon(y)+2\mu(\lambda+\mu)(\lambda+2\mu)^{-1}\mathrm{H}_3(y)\nabla\nabla\cdot\nabla w(y)+$$
$$+\tfrac{1}{2}\mathrm{H}_1(y)[-Q(H_+-H_-;y,\nabla)\upsilon(y)+\tfrac{1}{2}Q(H_+^2+H_-^2;y,\nabla)\nabla w(y)]\}=0$$
$$\mathrm{H}_3=\tfrac{1}{3}(H_+^3+H_-^3) \tag{2.9}$$

Thus, we have constructed the system of equations (2.7), (2.9) in K, which must be satisfied by the leading terms υ and w of series (2.1). We shall now remove the small dummy parameter by setting $\varepsilon=1$ and $x=(y,\ \zeta)$ in (1.3). It will be verified below that the terms of the series subsequent to U^0 and U^1 are small, because they decrease rapidly as $r\to 0$.

The resulting system is not formally self-adjoint. However, this deficiency can be easily corrected. We multiply (2.7) by $\tfrac{1}{2}(H_+-H_-)$, apply the operator $\nabla\cdot$, and add the result to (2.9).

The original system is transformed into the equivalent self-adjoint system

$$-Q(\mathrm{H}_1; y, \nabla)\upsilon(y) + Q(\mathrm{H}_2; y, \nabla)\nabla w(y) = 0 \tag{2.10}$$

$$\nabla \cdot \{-Q(\mathrm{H}_2; y, \nabla)\upsilon(y) + Q(\mathrm{H}_3; y, \nabla)\nabla w(y)\} = 0 \tag{2.11}$$

of partial differential equations. We shall write (2.10) and (2.11) in the short form

$$T^{11}(y, \nabla)\upsilon(y) + T^{12}(y, \nabla)w(y) = 0$$

$$T^{21}(y, \nabla)\upsilon(y) + T^{22}(y, \nabla)w(y) = 0 \tag{2.12}$$

where T^{11} is a (2×2)-matrix, T^{12} is a column, T^{21} is a row, and T^{22} is a scalar differential operator.

3. THE POINT OF CONTACT OF SMOOTH SURFACES

If K is the punctured plane $\mathbf{R}^2\setminus 0$ (the total angle), then the same system of equations (2.12) defines possible "beginnings" (υ, w) of the asymptotic solutions $\mathbf{u}$. The coefficients of the differential operators T^{jk} have strong degeneracy of order $r^{2m(j+k-1)}$ at O. Because of this, the general results of [10–12] cannot be applied directly. However, on multiplying (2.10) and (2.11) by r^{-2m} and r^{-4m}, respectively, and making the substitution $w \mapsto w' = r^{2m}w$, the elements $S^{jk}(y, \nabla)$ of the differential operator obtained in this way can be written as follows:

$$\begin{aligned} S^{11}(y, \nabla) &= r^{-2}\mathbf{S}^{11}(\varphi, \partial_\varphi, r\partial_r), & S^{12}(y, \nabla) &= r^{-3}\mathbf{S}^{12}(\varphi, \partial_\varphi, r\,\partial_r) \\ S^{21}(y, \nabla) &= r^{-3}\mathbf{S}^{21}(\varphi, \partial_\varphi, r\partial_r), & S^{22}(y, \nabla) &= r^{-4}\mathbf{S}^{22}(\varphi, \partial_\varphi, r\,\partial_r) \end{aligned} \tag{3.1}$$

The singularities of the coefficients in (3.1) are consistent with the order of the differential operators and are admissible in the theory of elliptic problems in domains with conic or corner points (see [10–12], etc.). It follows, in particular, that there is a denumerable system of linearly independent power solutions of system (2.12) on $\mathbf{R}^2\setminus 0$ (similar to (1.2)). These solutions can be represented in the form

$$\upsilon(y) = r^{\Lambda+1}\sum_{k=0}^{q}\frac{1}{k!}(\ln r)^k\Phi^{(q-k)}(\varphi)$$

$$w(y) = r^{\Lambda+2-2m}\sum_{k=0}^{q}\frac{1}{k!}(\ln r)^k\Phi_3^{(q-k)}(\varphi) \tag{3.2}$$

Additional information about system (2.12) is needed to explain how to find the numbers Λ and the sequences $\mathbf{\Phi}^{(0)} = (\Phi^{(0)}, \Phi_3^{(0)})$, $\mathbf{\Phi}^{(1)} = (\Phi^{(1)}, \Phi_3^{(1)}), \ldots$ in (3.2).

Let υ_r and υ_φ be the polar components of υ. We will write the matrix-valued operator T in (2.12) in terms of the polar coordinates r, φ. To do this, by (2.7) and (2.10), (2.11), we will change to polar coordinates in the equilibrium equations (we transform Q), in which $\mathbf{H}_i\tau(\upsilon)$ and $\mathbf{H}_j\tau(w)$ appear as the stress tensors. As a result, system (2.12) takes the form

$$\begin{aligned} &r^{2m-2}\mathbf{T}^{11}(\varphi,\partial_\varphi,r\partial_r)(\upsilon_r,\upsilon_\varphi) + r^{4m-3}\mathbf{T}^{12}(\varphi,\partial_\varphi,r\partial_r)w = 0 \\ &r^{4m-3}\mathbf{T}^{21}(\varphi,\partial_\varphi,r\partial_r)(\upsilon_r,\upsilon_\varphi) + r^{6m-4}\mathbf{T}^{22}(\varphi,\partial_\varphi,r\partial_r)w = 0 \end{aligned} \tag{3.3}$$

On the unit circle S_1^1 we consider the following operator pencil $\Lambda \mapsto \mathbf{T}(\varphi, \partial_\varphi, \Lambda)$ which depends polynomially on Λ

$$\mathbf{T}(\varphi,\partial_\varphi,\Lambda)=\begin{Vmatrix}\mathbf{T}^{11}(\varphi,\partial_\varphi,\Lambda+1) & \mathbf{T}^{12}(\varphi,\partial_\varphi,\Lambda+2-2m)\\ \mathbf{T}^{21}(\varphi,\partial_\varphi,\Lambda+1) & \mathbf{T}^{22}(\varphi,\partial_\varphi,\Lambda+2-2m)\end{Vmatrix} \tag{3.4}$$

It can be verified (see [12, Sec. 3.5]) that the functions (3.2) satisfy (3.3) if and only if Λ is an eigenvalue of (3.4) on S_1^1 and $\Phi^{(0)}, \dots, \Phi^{(q)}$ form a Jordan chain corresponding to this eigenvalue (i.e. $\Phi^{(0)}$ is an eigenvector and $\Phi^{(1)}, \dots, \Phi^{(q)}$ are the associated vectors).

Green's formula

$$\begin{gathered}\langle T^{11}\upsilon^1+T^{12}w^1,\upsilon^2\rangle_K+\langle T^{21}\upsilon^1+T^{22}w^1,w^2\rangle_K=2E(\upsilon^1,w^1;\upsilon^2,w^2)\\ 2E(\upsilon^1,w^1;\upsilon^2,w^2)=W(\upsilon^1,\upsilon^2;\mathrm{H}_1)-W(\nabla w^1,\upsilon^2;\mathrm{H}_2)-W(\upsilon^1,\nabla w^2;\mathrm{H}_2)+W(\nabla w^1,\nabla w^2;\mathrm{H}_3)\\ W(\upsilon^1,\upsilon^2;\mathrm{H})=\frac{1}{2\mu}\sum_{i,j=1}^{2}\left\{\langle \mathrm{H}\tau_{ij}(\upsilon^1),\tau_{ij}(\upsilon^2)\rangle_K-\frac{\lambda}{2(\lambda+\mu)}\langle \mathrm{H}\tau_{ii}(\upsilon^1),\tau_{jj}(\upsilon^2)\rangle_K\right\}\end{gathered} \tag{3.5}$$

holds for arbitrary vector-valued functions $(\upsilon^i, w^i)\in C_0^\infty(K)^3$. We denote by $\langle\, ,\,\rangle_K$ the scalar product in $L_2(K)^3$.

From (3.5) we obtain Green's formula

$$\langle T(\upsilon^1 w^1),(\upsilon^2,w^2)\rangle_K=\langle(\upsilon^1 w^1),T(\upsilon^2,w^2)\rangle_K \tag{3.6}$$

We write this equality in terms of the polar coordinates r, φ choosing

$$\begin{gathered}(\upsilon^1(y),w^1(y))=(r^{\Lambda+1}\Phi(\varphi),r^{\Lambda+2-2m}\Phi_3(\varphi))\\ (\upsilon^2(y),w^2(y))=(\chi_\delta(r)r^{\mathrm{M}+1}\Psi(\varphi),\chi_\delta(r)r^{\mathrm{M}+2-2m}\Psi_3(\varphi))\\ \mathrm{M}=-\overline{\Lambda}-2-2m\end{gathered} \tag{3.7}$$

as (υ^i, w^i). Here χ_δ is the cut-off function in $C_0^\infty(\mathbf{R})$ such that $\chi_\delta(r)=1$ for $\delta<r<\delta^{-1}$, and $\chi_\delta(r)=0$ for $r>2\delta^{-1}$ or $r<\delta/2$. We have

$$\begin{aligned}&|2\ln\delta|^{-1}\int_K\{(r^{2m-2}\mathrm{T}^{11}(r\partial_r)r^{1+\Lambda}\Phi+r^{4m-3}\mathrm{T}^{12}(r\partial_r)r^{2-2m+\Lambda}\Phi_3)\chi_\delta r^{1+\overline{\mathrm{M}}}\overline{\Psi}+\\ &+(r^{4m-3}\mathrm{T}^{21}(r\partial_r)r^{1+\Lambda}\Phi+r^{6m-4}\mathrm{T}^{22}(r\partial_r)r^{2-2m+\Lambda}\Phi_3)\chi_\delta r^{2-2m+\overline{\mathrm{M}}}\overline{\Psi}_3\}dy=\\ &=|2\ln\delta|^{-1}\int_\delta^{\delta^{-1}}\int_0^{2\pi}\{(\mathrm{T}^{11}(1+\Lambda)\Phi+\mathrm{T}^{12}(2-2m+\Lambda)\Phi_3\overline{\Psi}+\\ &+(\mathrm{T}^{21}(1+\Lambda)\Phi+\mathrm{T}^{22}(2-2m+\Lambda)\Phi_3)\overline{\Psi}_3\}r^{-1}drd\varphi+o(1)=\\ &=|2\ln\delta|^{-1}\int_\delta^{\delta^{-1}}\frac{dr}{r}\int_0^{2\pi}\{(\mathrm{T}^{11}(-\Lambda-1-2m)\overline{\Psi}+\mathrm{T}^{12}(-\Lambda-4m)\overline{\Psi}_3)\Phi+\\ &+(\mathrm{T}^{21}(-\Lambda-1-2m)\overline{\Psi}+\mathrm{T}^{22}(-\Lambda-4m)\overline{\Psi}_3)\Phi_3\}d\varphi\end{aligned}$$

To simplify the notation, we will omit the arguments φ and ∂_φ. Passing to the limit as $\delta\to 0$, we obtain

$$\langle\mathbf{T}(\Lambda)\Phi,\Psi\rangle_{S_1^1}=\langle\Phi,\mathbf{T}(-\overline{\Lambda}-2-2m)\Psi\rangle_{S_1^1} \tag{3.8}$$

It follows that the operator $\mathbf{T}(\varphi,\ \partial_\varphi,\ -\overline{\Lambda}-2-2m)$ is formally adjoint to $\mathbf{T}(\varphi,\ \partial_\varphi,\ \Lambda)$. The spectrum of (3.4) in the complex plane is therefore symmetric about the straight line $\{\Lambda\in\mathbf{C},\ \operatorname{Re}\Lambda=-1-m\}$. Besides, since the original operator T is elliptic, the spectrum of the operator pencil consists of a denumerable number of normal eigenvalues, all of which (except perhaps a finite number) lie in $\{\Lambda\in\mathbf{C}:\ \operatorname{Im}\Lambda<k\operatorname{Re}\Lambda\}$, where $k>0$ [10, 13]. Since the energy E in (3.5) vanishes only for polynomials, no points of the spectrum lie on the above-mentioned straight line ([12, Sec. 5.4]), which implies that $\{\Lambda \in \mathbf{C}: |\operatorname{Re}\Lambda+1+m|<\delta\}$ does not contain any eigenvalues.

The problem of finding all eigenvalues and Jordan chains of the operator pencil (3.4) involves solving a non-self-adjoint system of differential equations with variable coefficients, which cannot be done explicitly. The six polynomials

$$(1,0,0),\quad (0,1,0),\quad (-y_2,y_1,0),\quad (0,0,1),\quad (0,0,y_1),\quad (0,0,y_2) \tag{3.9}$$

are the exceptions. They will be denoted by $\mathbf{Z}^j$ $(j=1,\ldots,6)$. Formula (2.3) relates rigid displacements with polynomials. The vector-valued functions (3.9) have the form (3.2), Λ being equal to -1, -1, 0, $2m-2$, $2m-1$, $2m-1$, respectively. By the symmetry of the spectrum, $-1-2m$, $-1-2m$, $-2-2m$, $-4m$, $-1-4m$, $-1-4m$ are also eigenvalues (repeated numbers indicate multiplicity). We will denote the corresponding eigenvectors by $\boldsymbol{\Psi}^j$ $(j=1,\ldots,6)$. Solutions of the form (3.7) with angular parts $\boldsymbol{\Psi}^j$ give rise to the asymptotic solutions $\mathbf{U}^j$, which cause the energy functional to become infinite. We will give a physical interpretation of such solutions.

In the special case when $H_+=H_-$, the operators $\mathbf{T}^{12}$ and $\mathbf{T}^{21}$ vanish and system (3.3) splits into a system for υ and an equation for w. This simplifies the problem of finding $\boldsymbol{\Psi}^j$. For example

$$(\Psi_r^1(\varphi),\Psi_\varphi^1(\varphi))=\{8\mu m\pi h(\lambda^*+2\mu+m\mu)\}^{-1}([\lambda^*+3\mu+2m\mu]\cos\varphi,$$

$$-[\lambda^*+3\mu-2m\lambda^*]\sin\varphi);\quad \Psi_3^1(\varphi)=0$$

$$(\Psi_r^2(\varphi),\Psi_\varphi^2(\varphi))=\{8\mu m\pi h(\lambda^*+2\mu+m\mu)\}^{-1}([\lambda^*+3\mu+2m\mu]\sin\varphi,$$

$$[\lambda^*+3\mu-2m\lambda^*]\cos\varphi);\quad \Psi_3^2(\varphi)=0$$

$$\Psi_r^4(\varphi)=\Psi_\varphi^4(\varphi)=0;\quad \Psi_3^4(\varphi)=[2(\lambda^*+2\mu)\pi m(3m-1)^2h^3]^{-1}$$

$$\lambda^*=(\lambda+2\mu)^{-1}2\lambda\mu;\quad h_+(\varphi)=h_-(\varphi)=\tfrac{1}{2}h(\varphi)$$

It can be verified that

$$-\lim_{\varepsilon\to 0}\left\{\varepsilon\int_{-H_-}^{H_+}\sigma^{(r)}(\mathbf{U}^j)\cdot\mathbf{Z}^k\Big|_{r=\varepsilon}d\varphi dz\right\}=\delta_{j,k}\qquad \sigma^{(r)}=(\sigma_{rr},\sigma_{r\varphi},\sigma_{rz})$$

It follows that the solutions $\mathbf{u}^1$, $\mathbf{u}^2$ correspond to the longitudinal forces (in the y_1 and y_2 directions) concentrated at O, while $\mathbf{u}^4$ corresponds to the transverse force (in the z direction). Similarly, $\mathbf{u}^3$, $\mathbf{u}^5$, $\mathbf{u}^6$ are the concentrated moments.

Let us discuss the general case. First we will consider system (2.12). Let $\boldsymbol{\Phi}^1,\ldots,\boldsymbol{\Phi}^J$ be eigenvectors and let Λ_0 be an eigenvalue of $\mathbf{T}(\Lambda)$, and let there be no associated vectors. The number $\mathbf{M}_0=-2-2m-\Lambda_0$ is also an eigenvalue and has no associated vectors. Moreover, by [11, 12] the eigenvectors $\boldsymbol{\Psi}^1,\ldots,\boldsymbol{\Psi}^J$ can be chosen in such a way that the biorthogonality and normalization conditions

$$\int_0^{2\pi}\partial_\Lambda\mathbf{T}(\varphi,\partial_\varphi,\Lambda_0)\boldsymbol{\Phi}^j(\varphi)\cdot\boldsymbol{\Psi}^k(\varphi)d\varphi=\delta_{j,k},\quad j,k=1,\ldots,J \tag{3.10}$$

are satisfied. (We remark that analogous conditions are given in [11, 12] in the case of non-trivial Jordan chains.) If $X^1=(\upsilon^1,w^1),\ldots,X^J=(\upsilon^J,w^J)$ and $Y^1=(V^1,W^1),\ldots,Y^J=(V^J,W^J)$ are solutions of the form (3.7) constructed from Λ_0, $\boldsymbol{\Psi}^1,\ldots,\boldsymbol{\Psi}^J$ and $\mathbf{M}_0$, $\boldsymbol{\Psi}^1,\ldots,\boldsymbol{\Psi}^J$, respectively, then conditions (3.10) are equivalent to the relations

$$\langle T_\chi X^j,Y^k\rangle_K-\langle \chi X^j,TY^k\rangle_K=\delta_{j,k},\quad j,k=1,\ldots,J \tag{3.11}$$

Here χ is the cut-off function from $C^\infty(\mathbf{R}^2)$ equal to unity near the origin. Finally, integrating by parts, we transform (3.11) to the form

$$\varepsilon^{-1}\sum_{l,i=1}^{2}\int_0^{2\pi} n_l\{[-\mathrm{H}_1\tau_{li}(\upsilon^j)+\mathrm{H}_2\tau_{li}(\nabla w^j)]V_i^k-[-\mathrm{H}_1\tau_{li}(V^k)+\mathrm{H}_2\tau_{li}(\nabla W^k)]\upsilon_i^j+$$
$$+[\mathrm{H}_2\tau_{li}(\upsilon^j)-\mathrm{H}_3\tau_{li}(\nabla w^j)]\partial_i W^k-[\mathrm{H}_2\tau_{li}(V^k)-\mathrm{H}_3\tau_{li}(\nabla W^k)]\partial_i w^j+$$
$$+\partial_i[-\mathrm{H}_2\tau_{li}(\upsilon^j)+\mathrm{H}_3\tau_{li}(\nabla w^j)]W^k-$$
$$-\partial_i[-\mathrm{H}_2\tau_{li}(V^k)+\mathrm{H}_3\tau_{li}(\nabla W^k)]w^j\}|_{r=\varepsilon}\,d\varphi=\delta_{j,k} \tag{3.12}$$
$$n=(n_1,n_2)=(\cos\varphi,\sin\varphi)$$

Note that the left-hand sides of (3.12) and (3.11) are independent of the choice of χ and $\varepsilon>0$, respectively.

We shall now show how (3.12) gives rise to similar normalization conditions for the asymptotic solutions $\mathbf{X}^j$ and $\mathbf{Y}^k$ constructed in accordance with (2.3) and (2.5), (2.8) from the solutions $\mathbf{X}^j$ and $\mathbf{Y}^k$. To do this we compute the integral over the domain $\Xi(\varepsilon)$ in Ω defined by the intersection with the cylindrical surface $\{x:r=\varepsilon\}$

$$\int_{\Xi(\varepsilon)}\{\sigma^{(r)}(\mathbf{X}^j)\cdot\mathbf{Y}^k-\sigma^{(r)}(\mathbf{Y}^k)\cdot\mathbf{X}^j\}ds_x \tag{3.13}$$
$$\Xi(\varepsilon)=\{x:r=\varepsilon,\varphi\in[0,2\pi),\quad -H_-(x_1,x_2)<x_3<H_+(x_1,x_2)\}$$

By (2.3), (2.5), and (2.8), it is equal to

$$\varepsilon^{-1}\sum_{l,i=1}^{2}\int_0^{2\pi}\int_{-H_-}^{H_+} n_l\{[\tau_{li}(\upsilon^j)-x_3\tau_{li}(\nabla w^j)](V_i^k-x_3\partial_i W^k)-$$
$$-[\tau_i(V^k)-x_3\tau_{li}(\nabla W^k)](\upsilon_i^j-x_3\partial_i w^j)+$$
$$+(\tfrac12\partial_i[(H_+-H_-)\tau_{li}(\upsilon^j)-\tfrac12(H_+^2+H_-^2)\tau_{li}(\nabla w^j)]-$$
$$-x_3\partial_i\tau_{li}(\upsilon^j)+\tfrac12 x_3^2\partial_i\tau_{li}(\nabla w^j))W^k-$$
$$-(\tfrac12\partial_i[(H_+-H_-)\tau_{li}(V^k)-\tfrac12(H_+^2+H_-^2)\tau_{li}(\nabla W^k)]-$$
$$-x_3\partial_i\tau_{li}(V^k)+\tfrac12 x_3^2\partial_i\tau_{li}(\nabla W^k))w^j\}|_{r=\varepsilon}\,d\varphi dx_3 \tag{3.14}$$

apart from terms $o(1)$. We integrate with respect to $x_3\in[-H_-,\ H_+]$, replace the vectors (υ^j, w^j) and $(V^k,\ W^k)$ by their representations (3.7) in polar coordinates, and add the following integral to the result, the integral being equal to zero by (2.7)

$$\varepsilon^{-1}\sum_{l,i=1}^{2}\int_0^{2\pi} n_l\tfrac12(H_+-H_-)\{\partial_i[-\mathrm{H}_1\tau_{li}(\upsilon^j)+\mathrm{H}_2\tau_{li}(\nabla w^j)]W^k-$$
$$-\partial_i[-\mathrm{H}_1\tau_{li}(V^k)+\mathrm{H}_2\tau_{li}(\nabla W^k)]w^j\}|_{r=\varepsilon}\,d\varphi$$

After these operations, the integral (3.14) becomes equal to the left-hand side of (3.12). It follows that (3.13) tends to the Kronecker delta $\delta_{j,k}$ as $\varepsilon\to 0$.

By the aforesaid, if the rigid displacements (3.9) are now chosen as $\mathbf{Y}^k$, then one can find a system of asymptotic solutions $\{\mathbf{U}^1,\ldots,\mathbf{U}^6\}$ such that

$$\lim_{\varepsilon\to 0}\int_{\Xi(\varepsilon)}\sigma^{(r)}(\mathbf{U}^p)\cdot\mathbf{Z}^q ds_x=\delta_{p,q},\quad p,q=1,\ldots 6$$

This equality makes it possible to interpret the displacement fields $\mathbf{U}^1,\ldots,\mathbf{U}^6$ as those corresponding to concentrated forces and moments.

We also remark that the above-mentioned biorthogonality and normalization conditions play an important role when the integral formulae for the coefficients are derived from the asymptotic expansions of the solutions of problems near the singularities of the boundary (see [11, 12], Chaps 3 and 4; similar formulae in the theory of cracks involve weight functions [14, 15]).

4. THE CASE OF A RIGIDLY SUPPORTED LATERAL SURFACE

If the Dirichlet conditions are specified on $\Sigma_\pm$, then the limiting system of equations (2.12) in K will be supplemented with the following boundary conditions

$$v_1(y) = v_2(y) = 0, \quad w(y) = 0, \quad \partial_\varphi w(y) = 0 \quad \text{for} \quad \varphi = \pm\alpha \tag{4.1}$$

Now, the operator pencil corresponding to problem (2.12), (4.1) consists of the differential operator (3.4) on the arc $(-\alpha, \alpha)$ and the boundary conditions at $\varphi = \pm\alpha$. In other words, to determine Λ and Φ_i in (3.2) it is necessary to solve the eigenvalue problem

$$\begin{gathered} \mathbf{T}(\varphi, \partial_\varphi, \Lambda)\mathbf{\Phi}(\varphi) = 0 \\ \Phi_1(\pm\alpha) = \Phi_2(\pm\alpha) = \Phi_3(\pm\alpha) = \partial_\varphi\Phi_3(\pm\alpha) = 0 \end{gathered} \tag{4.2}$$

The corresponding operator pencil retains all the properties of (3.4) (except that the vectors (3.9) are no longer solutions of the boundary-value problem (2.12), (4.1)).

We shall state an assertion on the asymptotic form of the solution of the problem in Ω, making (for brevity) a number of simplifying assumptions. Suppose that $\{\Lambda \in \mathbf{C}:\ 0 \leqslant \operatorname{Re}\Lambda + 1 + m < \delta_1\}$ contains one eigenvalue Λ_0 of problem (4.2), which, in addition, is a simple eigenvalue. Let $\mathbf{\Phi}^0$ be the eigenvector corresponding to Λ_0 and let $\mathbf{u}^0$ be a segment of the asymptotic series (2.3), in which

$$v^0(y) = r^{\Lambda_0+1}(\Phi_1^0(\varphi), \Phi_2^0(\varphi)), \quad w^0(y) = r^{\Lambda_0+2-2m}\Phi_3^0(\varphi),$$

$$\mathbf{u}^0(x) = \sum_{j=0}^{3} U^j(x) \tag{4.3}$$

Then the energy solution of the problem on the deformation of Ω by loads applied away from O has the asymptotic representation

$$\begin{gathered} u(x) = c_0\mathbf{u}^0(x) + o(r^\delta) \times (r, r, r^{2-2m}), \quad r \to 0 \\ \operatorname{Re}\Lambda_0 < \delta < \min\{\delta_1, 2m - 1 + \operatorname{Re}\Lambda_0\} \end{gathered} \tag{4.4}$$

In (4.4) c_0 is a constant depending on the data of the problem as a whole (on the loads and the geometry of the domain). It is clear why δ_1 occurs in the bounds for δ: the asymptotic form of u contains the asymptotic solution $\mathbf{u}^1$ corresponding to the eigenvalue Λ_1 on the straight line $\{\Lambda:\ \operatorname{Re}\Lambda = -1 - m + \delta_1\}$. The inequality appears because (4.3) does not involve any solutions of the boundary-layer type. The boundary layer compensates the discrepancies left by the solution $\mathbf{u}^0$ in the boundary conditions on $\Sigma_\pm$, and decays exponentially away from $\Sigma_\pm$. We shall show how to construct the boundary layer, and what is the result of taking it into account in the simplified symmetric situation when

$$H_+ = H_- = \tfrac{1}{2}H$$

The scheme for finding the boundary layer is the same as in the theory of thin plates. Consider one of the lateral surfaces of the beak, say Σ_- to be specific. By analogy with (1.3), introducing a small dummy parameter ε, we define the special coordinates

$$\xi_1 = \varepsilon^{\gamma-1}\mathbf{H}(\xi_3)^{-1}(y_1 \sin\alpha + y_2 \cos\alpha), \quad \xi_2 = \mathbf{H}(\xi_3)^{-1}\zeta$$

$$\xi_3 = y_1 \cos\alpha - y_2 \sin\alpha \quad (\mathbf{H}(r) = H(r\cos\alpha, r\sin\alpha)) \tag{4.5}$$

in the neighbourhood of Σ_-.

Because of the additional factor $\varepsilon^{\gamma-1}$ in the first formula in (4.5) (expansion in the normal direction to Σ_-), $\xi^0 = (\xi_1, \xi_2)$ are now the "fast" variables. They form Cartesian coordinates in planes normal to Σ_-. The variable ξ_1 must be regarded as a slow one. This corresponds to a new splitting of the Lamé operator

$$L(\nabla_x) = \varepsilon^{-2}\mathbf{H}(\xi_3)^{-2}\mathbf{L}^0(\nabla^0) + \dots \tag{4.6}$$

Here $\nabla_0 = (\partial/\partial\xi_1, \partial/\partial\xi_2)$, the dots denote lower-order terms, and $\mathbf{L}_0$ is a 3×3 block-diagonal operator. The two-dimensional Lamé operator with constants λ and μ serves as the upper block of $\mathbf{L}^0$, while the Laplace operator $\mu\nabla^0\cdot\nabla^0$ plays the role of the right lower element. The operator $B^{\pm}(x, \nabla_x)$ of the boundary conditions on $\Gamma_\pm$ has a similar structure with the leading part $\varepsilon^{-1}\mathbf{H}(\xi_3)^{-1}\mathbf{B}^{0\pm}(\nabla^0)$. Finally, in the neighbourhood of Σ_-, Ω is given by the inequalities

$$\xi_1 > 0, \quad |\xi_2| < {}^1\!/\!{}_2\mathbf{H}(\xi_3)^{-1}H(y), \quad \xi_3 > 0 \tag{4.7}$$

Since, by (4.5), $H(y) = \mathbf{H}(\xi_3) + O(\varepsilon^{1-\gamma})$ near Σ_- by taking (4.7) and (4.6) into account, we find that the formal passage to $\varepsilon = 0$ gives rise to the following limiting problem in $\Pi = \{\xi^0 \in \mathbf{R}^2 : \xi_1 > 0, \ |\xi_2| < \tfrac{1}{2}\}$ depending on ξ_3

$$\mathbf{L}^0(\nabla^0)\mathbf{V}(\xi) = \mathbf{F}(\xi), \quad \xi^0 \in \Pi \tag{4.8}$$

$$\mathbf{B}^{0\pm}(\nabla^0)\mathbf{V}(\xi_1, \pm{}^1\!/\!{}_2, \xi_3) = \mathbf{G}^{\pm}(\xi_1, \xi_3), \quad \xi_1 > 0 \tag{4.9}$$

$$\mathbf{V}(0, \xi_2, \xi_3) = \mathbf{G}^0(\xi_2, \xi_3), \quad |\xi_2| < {}^1\!/\!{}_2 \tag{4.10}$$

No mass forces $\mathbf{F}$ are present in the problem for the leading terms of the boundary layer, and the lateral surfaces of Π are free of any loads, i.e. the homogeneous equality (4.9) means that the stresses $\boldsymbol{\sigma}_{j2}(\mathbf{V}; \xi_1, \pm\tfrac{1}{2})$ vanish for $j = 1, 2, 3$ and

$$\boldsymbol{\sigma}_{33}(\mathbf{V}) = \lambda(\partial \mathbf{V}_1/\partial\xi_1 + \partial \mathbf{V}_2/\partial\xi_2), \quad \boldsymbol{\sigma}_{3k}(\mathbf{V}) = \boldsymbol{\sigma}_{k3}(\mathbf{V}) = \mu\partial\mathbf{V}_3/\partial\xi_k$$

$$\boldsymbol{\sigma}_{jk}(\mathbf{V}) = \mu(\partial\mathbf{V}_j/\partial\xi_k + \partial\mathbf{V}_k/\partial\xi_j) + \delta_{j,k}\lambda(\partial\mathbf{V}_1/\partial\xi_1 + \partial\mathbf{V}_2/\partial\xi_2), \quad j, k = 1, 2$$

If $H_+ = H_-$ (a symmetric beak) system (2.12) breaks up: Eqs (2.10) and (2.11) take the form $T^{11}\upsilon = 0$ and $T^{22}w = 0$. Let $w = 0$ initially. Since the equalities (4.1) are satisfied, the leading term of the discrepancy of series (2.3) is given by

$$U^2(y, \zeta) = (\mathbf{0}, -\lambda[\lambda + 2\mu]^{-1}\zeta\nabla\cdot\upsilon(y)) \tag{4.11}$$

under the boundary conditions on $\Sigma_\pm$.

It follows that to compensate the discrepancy, one must solve problem (4.8)–(4.10) with right-hand sides

$$\mathbf{F} = 0, \quad \mathbf{G}^{\pm} = 0, \quad \mathbf{G}^0(\xi_2, \xi_3) = \mathbf{H}(\xi_3)\lambda(\lambda + 2\mu)^{-1}\xi_2\nabla\cdot\upsilon(\xi_3)\mathbf{e}^2 \tag{4.12}$$

where $\mathbf{e}^j$ is the unit vector of the ξ_j axis. The solution $\mathbf{V}$ of this problem has the following asymptotic form at infinity [16]

$$\mathbf{V}(\xi) = \mathbf{H}(\xi_3)c_1(\nu)\nabla\cdot\upsilon(\xi_3)\mathbf{e}^1 + o(\exp(-\delta_0\xi_1)), \quad \xi_1 \to \infty \tag{4.13}$$

Here $\delta_0 > 0$ and $c_1(\nu)$ depends only on Poisson's ratio $\nu = \lambda[2(\lambda+\mu)]^{-1}$ (its graph is given in

[16, p. 20]). Since, for $\nu>0$, the solution does not vanish at infinity, the basic requirement for the boundary layer is violated. The situation can be corrected by altering the form of (4.11)

$$U^2(y,\zeta)=(\upsilon^1(y),-\lambda[\lambda+2\mu]^{-1}\zeta\nabla\cdot\upsilon^0(y)) \tag{4.14}$$

Here $\upsilon^0\equiv\upsilon$, and $\upsilon^1=(\upsilon_1^1,\ \upsilon_2^1)$ is a vector-valued function to be determined. The form of the discrepancy is also altered: the term $-\upsilon_\varphi^1(\xi_3)\mathbf{e}^1-\upsilon_r^1(\xi_3)\mathbf{e}^3$ is added to the last expression in (4.12). The sum $\mathbf{V}^1$ of this term and the previous solution $\mathbf{V}$ is a solution of the problem (4.8)–(4.10) with a new right-hand side. By (4.13)

$$\mathbf{V}^1(\xi)=\{\mathbf{H}(\xi_3)c_1(\nu)\nabla\cdot\upsilon^0(\xi_3)-\upsilon_\varphi^1(\xi_3)\}\mathbf{e}^1+\upsilon_r^1(\xi_3)\mathbf{e}^3+o(\exp(-\delta_0\varepsilon_1)),\quad \xi_1\to\infty$$

Now, the exponential decay condition for the boundary layer furnishes the following boundary condition for the unknown vector υ^1

$$\upsilon_r^1(y)=0,\quad \upsilon_\varphi^1(y)=H(y)c_1(\nu)\nabla\cdot\upsilon^0(y)\quad \text{for}\quad \varphi=\pm\alpha \tag{4.15}$$

In (4.15) we also state a condition, which can be obtained by considering the boundary layer near Σ_+. The system of equations for υ^1 has the form

$$T^{11}(y,\nabla)\upsilon^1(y)=0,\qquad y\in K \tag{4.16}$$

It occurs as the condition for problem (2.4) to be solvable with $j=4$, which arises additionally in connection with the modification (4.14) of the term (4.11) of series (2.3) (we recall that a similar system for υ^0 arises when problem (2.4) with $j=3$ is solved). Thus, we have obtained a problem in K with a right-hand side of special form, the solution of which can be found as in [12, Sec. 3.5]. If Λ_0+2m-1 is not an eigenvalue of the operator pencil, then

$$\upsilon^1(y)=r^{\Lambda_0+2m}\Upsilon^1(\varphi) \tag{4.17}$$

The angular part Υ^1 can be found from the problem

$$\mathbf{T}(\gamma,\partial_\varphi,\Lambda_0+2m-1)\Upsilon^1(\varphi)=0,\qquad \varphi\in(-\alpha,\alpha)$$
$$\Upsilon_r^1(\pm\alpha)=0,\quad \Upsilon_\varphi^1(\pm\alpha)=h(\pm\alpha)c_1(\nu)\partial_\varphi\Phi_\varphi^0(\pm\alpha)$$

But if Λ_0+2m-1 is an eigenvalue, then the factor Υ^1 in (4.17) becomes a polynomial of $\ln r$ (cf. (3.2)).

By analogy with the construction of the correction υ^1 in the asymptotic solution corresponding to the longitudinal deformation of a symmetric beak, one can compute the corrections w^1 and w^2 in the "bending" solution, which we represent as follows:

$$\begin{aligned}&(\mathbf{0},w^0(y))+\varepsilon^{1-\gamma}(\zeta\nabla w^0(y),w^1(y))+\\&+\varepsilon^{2-2\gamma}(\zeta\nabla w^1,w^2+{}^1\!/\!_2\lambda[\lambda+2\mu]^{-1}\zeta^2\nabla\cdot\nabla w^0(y))\end{aligned} \tag{4.18}$$

Here $w^0\equiv w$ and

$$w^1(y)=r^{\Lambda+1}\Upsilon_3^1(\varphi),\quad w^2(y)=r^{\Lambda+2m}\Upsilon_3^2(\varphi) \tag{4.19}$$

assuming that Λ_0+2m-1 and Λ_0+4m-2 are not eigenvalues of the above-mentioned operator pencil.

By (4.1), the first two terms in (4.18) satisfy the Dirichlet boundary condition on $\Gamma_\pm$ if

$$w^1(y)=0\quad \text{for}\quad \varphi=\pm\alpha \tag{4.20}$$

To compensate for the discrepancy of the third term we invoke the solution $\mathbf{V}^2$ of the boundary-layer type, i.e. the solution of problem (4.8)–(4.10) with right-hand sides

$$\mathbf{F}=0, \quad \mathbf{G}^{\pm}=0, \quad \mathbf{G}^0(\xi_2,\xi_3)=\partial_1 w^1(\xi_3)\mathbf{H}(\xi_3)\xi_2\mathbf{e}^1 - \\ -w^2(\xi_3)\mathbf{e}^2 - \tfrac{1}{2}\mathbf{H}(\xi_3)^2\lambda(\lambda+2\mu)^{-1}\nabla\cdot\nabla w^0(\xi_3)\xi_2^2\mathbf{e}^2$$

Using the notation $c(\nu)\equiv c_2(\nu)$ and $b(\nu)$ (see [17, p. 644], where a graph of $\nu\mapsto c(\nu)$ is also given), we compute the asymptotic form

$$\mathbf{V}^2(\xi)=[\partial_1 w^1(\xi_3)\mathbf{H}(\xi_3)\xi_2 + c_2(\nu)\mathbf{H}(\xi_3)^2\nabla\cdot\nabla w^0(\xi_3)\xi_2]\mathbf{e}^1 + \\ +[-w^2(\xi_3)+(b(\nu)-c_2(\nu)\xi_1)\nabla\cdot\nabla w^0(\xi_3)\mathbf{H}(\xi_3)^2]\mathbf{e}^2 + o(\exp(-\sigma_0\xi_1)), \quad \xi_1\to+\infty$$

On requiring that $\mathbf{V}^2$ should vanish exponentially as $\xi_1\to+\infty$, we arrive at the equalities

$$r^{-1}\partial_\varphi w^1(y) = -c_2(\nu)H(y)\partial_\varphi^2 w^0(y) \\ w^2(y)=b(\nu)H(y)^2\partial_\varphi^2 w^0(y) \tag{4.21}$$

Thus, w^1 in (4.18) can be found as a solution of the problem involving the boundary conditions (4.20) and (4.21) as well as the equation

$$T^{22}(y,\nabla)w^1(y)=0, \quad y\in K$$

This equation represents the condition for problem (2.4) to be solvable with $j=4$. The angular part Υ_3^1 in the representation (4.19) for w^1 can be found from the relations

$$\mathbf{T}^{22}(\varphi,\partial_\varphi,\Lambda_0+1)\Upsilon_3^1(\varphi)=0, \quad \varphi\in(-\alpha,\alpha) \\ \Upsilon_3^1(\pm\alpha)=0, \quad \partial_\varphi\Upsilon_3^1(\pm\alpha) = -h(\pm\alpha)c_2(\nu)\partial_\varphi^2\Phi_3^0(\pm\alpha)$$

Finally, we remark that the missing equation and boundary condition for w^2 arise at the next step of the algorithm (the solvability of (2.4) for $j=6$ and the requirement that the corresponding term of the boundary layer should decrease exponentially).

5. STRESS-FREE LATERAL SURFACES

Suppose that the stresses $\sigma_{r\varphi}$, $\sigma_{\varphi\varphi}$, and $\sigma_{\varphi z}$ are equal to zero on $\Sigma_\pm$ near the beak tip. We shall construct the boundary layers arising near the lateral surfaces, and, from the requirement that the boundary layers should decay exponentially, we shall determine the boundary conditions for the system of equations (2.12) for the vector-valued functions (υ, w). The terms of the series (2.3) leave discrepancies in the boundary conditions on $\Sigma_\pm$. In accordance with (2.5) and (2.8), the leading terms of these discrepancies are computed to be

$$\sigma_{r\varphi}(\mathbf{u})\sim\tau_{\nu s}(\upsilon)-x_3\tau_{\nu s}(\nabla w) \\ \sigma_{\varphi\varphi}(\mathbf{u})\sim\tau_{\nu\nu}(\upsilon)-x_3\tau_{\nu\nu}(\nabla w) \\ \sigma_{z\varphi}(\mathbf{u})\sim\sum_{i=\nu,s}\partial_i\{\tfrac{1}{2}(H_+-H_-)\tau_{\nu i}(\upsilon)-x_3\tau_{\nu i}(\upsilon)- \\ -\tfrac{1}{4}(H_+^2+H_-^2)\tau_{\nu i}(\nabla w)+\tfrac{1}{2}x_3^2\tau_{\nu i}(\nabla w)\} \tag{5.1}$$

Here $\tau_{s\nu}$ and $\tau_{\nu\nu}$ are the components of the stress tensor (for a plane stress state) written in terms of the local coordinates ν, s introduced near ∂K. Note that $\xi_2=\nu(\varepsilon\mathbf{H}(\xi_3))^{-1}$ and $\xi_3=s$.

We consider the surface Σ_- (to fix our ideas) and change to the fast variables (4.5) in the neighbourhood of that surface. By analogy with (4.7), Ω can be defined by

$$\xi_1 > 0, \quad \xi_2 \in [-\mathbf{H}(\xi_3)^{-1}H_-(y),\ \mathbf{H}(\xi_3)H_+(y)], \quad \xi_3 > 0$$

where $\mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-$ and, similarly as in (4.5), $\mathbf{H}_\pm(r) = (r\cos\alpha,\ r\sin\alpha)$. Thus, if we set $\varepsilon = 0$, then Ω becomes the half-layer $\{\xi \in \mathbf{R}^3,\ \xi_3 > 0,\ \xi^0 \in \Pi\}$, where

$$\Pi = \Upsilon \times \mathbf{R}^1_+, \quad \Upsilon = [-\mathbf{H}(\xi_3)^{-1}\mathbf{H}_-(\xi_3),\ \mathbf{H}(\xi_3)^{-1}\mathbf{H}_+(\xi_3)] \tag{5.2}$$

We note two facts: firstly, the interval Υ in (5.2) is independent of ξ_3 in view of (1.1). Secondly, the semi-strip becomes asymmetric, because the condition $H_\pm = \frac{1}{2}H$ introduced in Sec. 4 has been dropped. The leading terms of the decompositions (4.6) of L and B remain the same as in Sec. 4. This means that, as before, we find that the leading term $\mathbf{V}$ of the boundary layer satisfies the homogeneous equations (4.8) and (4.9) in the "new" strip (5.2) as well as the boundary conditions

$$\begin{aligned} &\mathbf{B}^{0\pm}(\nabla^0)\mathbf{V}(0,\xi_2,\xi_3) = \mathbf{G}^0(\xi_2,\xi_3), \quad \xi_2 \in \Upsilon \\ &\mathbf{G}^0(\xi_2,\xi_3) = \{-\tau_{vv}(\upsilon;\xi_3) + \xi_2\mathbf{H}(\xi_3)\tau_{vv}(\nabla w;\xi_3)\}e^1 + \\ &+\{-\tau_{vs}(\upsilon;\xi_3) + \xi_2\mathbf{H}(\xi_3)\tau_{vs}(\nabla w;\xi_3)\}e^3 \end{aligned} \tag{5.3}$$

The equalities

$$\int_\Upsilon \mathbf{G}^0_j(\xi_2,\xi_3)d\xi_2 = 0, \quad j = 1,2,3, \qquad \int_\Upsilon -\xi_2\mathbf{G}^0_1(\xi_2,\xi_3) = 0 \tag{5.4}$$

which mean that the principal vector and the principal moment of the load both vanish, serve as the conditions for such a problem to be solvable in the class of functions vanishing at infinity.

We will consider the first $(j = 1)$ and last equalities in (5.4). We compute the integrals and remove the dummy parameter by setting $\varepsilon = 1$. Since $\mathbf{H}_\pm(\xi_3) = H_\pm(y)$ for $y \in \partial K$, the resulting relations take the form

$$-\mathrm{H}_1(y)\tau_{vv}(\upsilon;\, y) + \mathrm{H}_2(y)\tau_{vv}(\nabla w;\, y) = 0 \tag{5.5}$$

$$\mathrm{H}_2(y)\tau_{vv}(\upsilon;\, y) - \mathrm{H}_3(y)\tau_{vv}(\nabla w;\, y) = 0, \quad y \in \partial K\backslash 0 \tag{5.6}$$

(here the boundary conditions arising in the study of the boundary layer near Σ_+ are also included). It follows from (5.6) and (5.5) that $\tau_{vv}(\upsilon) = \tau_{vv}(\nabla W) = 0$ for $\varphi = \pm\alpha$, i.e. $\mathbf{G}^0 \equiv 0$. Since the first pairs of rows in (4.8), (4.9) and (5.3) constitute the plane problem of the theory of elasticity for the vector $(\mathbf{V}_1,\ \mathbf{V}_2)$ in a half-strip, the aforesaid means that $\mathbf{V}_1 = \mathbf{V}_2 = 0$. The relation in (5.3) corresponding to $j = 3$ furnishes one more boundary condition

$$-\mathrm{H}_1(y)\tau_{vs}(\upsilon;\, y) + \mathrm{H}_2(y)\tau_{vs}(\nabla w;\, y) = 0, \quad y \in \partial K\backslash 0 \tag{5.7}$$

Moreover, $\mathbf{V}_3$ is non-zero: it is the solution of the following Neumann problem that vanishes at infinity

$$\begin{aligned} &\nabla^0 \cdot \nabla^0\mathbf{V}_3(\xi) = 0, \quad \xi^0 \in \Pi \\ &\mu\partial\mathbf{V}_3(\xi)/\partial\xi_2 = 0, \quad \xi_1 > 0, \quad \xi_2 = \pm\,\mathbf{H}(\xi_3)^{-1}\mathbf{H}_\pm(\xi_3) \\ &\mu\partial\mathbf{V}_3(\xi)/\partial\xi_1 = -\tau_{vs}(\upsilon;\,\xi_3) + \xi_2\mathbf{H}(\xi_3)\tau_{\varpi s}(\nabla w;\,\xi_3), \quad \xi_1 = 0,\ \xi_2 \in \Upsilon \end{aligned} \tag{5.8}$$

We remark that, by (5.7), the right-hand side of (5.8) is equal to

$$\mathbf{A}(\xi_3)(\xi_2 - \tfrac{1}{2}\mathbf{H}(\xi_3)^{-1}[\mathbf{H}_+(\xi_3) - \mathbf{H}_-(\xi_3)])$$
$$\mathbf{A}(\xi_3) = \mathbf{H}(\xi_3)\tau_{vs}(\nabla w;\ \xi_3).$$

System (2.1) requires four boundary conditions, but only three have been found so far. The point is that the component $\mathbf{G}_2^0$ turns out to be zero and the formula with $j=2$ in (5.4) does not provide the fourth condition. One must, therefore, resort to the next term $\varepsilon\mathbf{H}(\xi_3)\mathbf{V}^1(\xi_3)$ of the boundary layer. The vector $\mathbf{V}^1$ satisfies problem (4.8), (4.9), (5.3) with certain right-hand sides $\mathbf{F}$, $\mathbf{G}^\pm$, and $\mathbf{G}^0$. Again, there are four conditions for the problem to be solvable in the class of decreasing vector-valued functions. However, only one of them is necessary

$$\int_\Pi \mathbf{F}_2(\xi)d\xi^0 - \sum_\pm \pm\int_0^\infty \mathbf{G}_2^\pm(\xi_1,\xi_3)d\xi_1 + \int_\Upsilon \mathbf{G}_2^0(\xi_2,\xi_3)d\xi_2 = 0 \tag{5.9}$$

(the second component of the principal load vector is equal to zero). We emphasize that the remaining three conditions can be used (as in Sec. 4) to determine the boundary data for $(\upsilon^1,\ w^1)$.

Let us refine formula (4.6). The second term of the decomposition of $L(\nabla_x)$ has the form $\varepsilon^{-1}\mathbf{H}(\xi_3)^{-1}\mathbf{L}^1(\partial/\partial\xi)$ with $\mathbf{L}^1$ being a block-antidiagonal operator with right upper (2×1)-block $(\lambda+\mu)D_3\nabla^0$ and left lower (1×2)-block $(\lambda+\mu)D_3\nabla^0$ (see the expression for D_3 below). Applying the operator to $\mathbf{V}_3\mathbf{e}^3 + \varepsilon\mathbf{H}(\xi_3)\mathbf{V}^1$, we find that

$$\mathbf{F}_2(\xi) = -(\lambda+\mu)\mathbf{H}(\xi_3)D_3\partial_2\mathbf{V}_3(\xi) \tag{5.10}$$

The two-term decomposition of the boundary condition operator furnishes the relation

$$\begin{gathered}\mathbf{G}_2^\pm(\xi_1,\xi_3) = \mp\mu\mathbf{H}'(\xi_3)\partial_2\mathbf{V}_3(\xi) - \lambda D_3\mathbf{V}_3(\xi)\\ D_3 = \partial/\partial\xi_3 - \mathbf{H}^{-1}\mathbf{H}'(\xi_1\partial_1 + \xi_2\partial_2),\qquad \partial_j = \partial/\partial\xi_j\quad (j=1,2)\end{gathered} \tag{5.11}$$

The derivative with respect to ξ_3 is denoted by a prime. Henceforth the argument ξ_3 will be omitted. Apart from a multiplier, $\mathbf{G}_2^0$ is identical with the leading term of the discrepancy left by the solution $\mathbf{u}$ in the boundary condition for $\sigma_{z\varphi}$ on Σ_+ (see (5.1))

$$\begin{aligned}\mathbf{G}_2^0 = \mathbf{H}\sum_{i=v,s}\partial_i\{&-\tfrac{1}{2}(\mathbf{H}_+ - \mathbf{H}_-)\tau_{vi}(\upsilon) + \xi_2\mathbf{H}\tau_{vi}(\upsilon) +\\ &+\tfrac{1}{4}(\mathbf{H}_+^2 + \mathbf{H}_-^2)\tau_{vi}(\nabla w) - \tfrac{1}{2}\xi_2^2\mathbf{H}^2\tau_{vi}(\nabla w)\}\end{aligned} \tag{5.12}$$

To evaluate the integrals in (5.9), we represent $\mathbf{V}_3$ as the product $\mathbf{A}P(\xi^0)$ ($\mathbf{A}$ is determined from (5.8)). By (5.10), we have

$$\begin{aligned}\int_\Pi \mathbf{F}_2(\xi)d\xi^0 &= -(\lambda+\mu)\int_\Pi (\mathbf{A}'\mathbf{H} - \mathbf{A}\mathbf{H}')\partial_2 P(\xi^0) - \mathbf{A}\mathbf{H}'\{\xi_1\partial_1\partial_2 P(\xi^0) +\\ &+\xi_2\partial_2\partial_2 P(\xi^0)\}d\xi^0 = -(\lambda+\mu)\sum_\pm \pm\int_0^\infty (\mathbf{A}'\mathbf{H} - \mathbf{A}\mathbf{H}')P(\xi^0) -\\ &-\mathbf{A}\mathbf{H}'\{\xi_1\partial_1 P(\xi^0) - P(\xi^0)\}d\xi_1 = -(\lambda+\mu)(\mathbf{A}\mathbf{H})'\sum_\pm \pm\int_0^\infty p(\xi^0)d\xi_1\end{aligned}$$

($\xi_2 = \pm\mathbf{H}^{-1}\mathbf{H}_\pm$ in the contour integrals, i.e. integration is over the bases of Π). The second term in (5.9) can be determined using (5.11)

$$\begin{aligned}\sum_\pm \pm\int_0^\infty \mathbf{G}_2^\pm(\xi_1)d\xi_1 = \sum_\pm \int_0^\infty &-\mathbf{H}\mathbf{H}'\mu\partial_2 P(\xi^0) \pm\\ &\pm\lambda\{\mathbf{A}'\mathbf{H}P(\xi^0) - \mathbf{A}\mathbf{H}'(\xi_1\partial_1 P(\xi^0) + \xi_2\partial_2 P(\xi^0))\}d\xi_1 =\end{aligned}$$

$$= \sum_{\pm} \pm\lambda \int_0^{\infty} \{\mathbf{A}'\mathbf{H}P(\xi^0) - \mathbf{A}\mathbf{H}'\xi_1\partial_1 P(\xi^0)\} d\xi_1 =$$

$$= \lambda(\mathbf{A}\mathbf{H})' \sum_{\pm} \pm \int_0^{\infty} P(\xi^0) d\xi_1 \text{ when } \xi_2 = \pm\mathbf{H}^{-1}\mathbf{H}_{\pm}$$

This means that the sum of the first two terms in (5.6) is

$$-\mu(\mathbf{A}(\xi_3)\mathbf{H}(\xi_3))' \sum_{\pm} \pm \int_0^{\infty} P(\xi_1, \pm\mathbf{H}^{-1}\mathbf{H}_{\pm}) d\xi_1 \tag{5.13}$$

and now one must find the value of the integral. This can be done using Green's formula. Consider the function $Y(\xi^0) = \xi_2$. Obviously, Y is a harmonic function satisfying the condition $\partial_2 Y = 1$ on the bases of the half-strip and the condition $\partial_1 Y = 0$ on its end. According to (5.8), P is also a harmonic function with $\partial_2 P = 0$ for $\xi_1 > 0$ and $\xi_2 = \pm\mathbf{H}^{-1}\mathbf{H}$, and $\partial_1 P = \mu^{-1}$ ($\xi_2 = \frac{1}{2}\mathbf{H}^{-1}[\mathbf{H}_+ = \mathbf{H}_-]$) for $\xi_1 = 0$ and $\xi_2 \in \Upsilon$. We use Green's formula

$$0 = \int_{\Pi} \{Y\nabla^0 \cdot \nabla^0 P - P\nabla^0 \cdot \nabla^0 Y\} d\xi^0 - \int_{\Upsilon} Y\partial_1 P d\xi_2 - \sum_{\pm} \pm \int_0^{\infty} P d\xi_1$$

which implies that

$$\sum_{\pm} \pm \int_0^{\infty} P(\xi_1, \pm\mathbf{H}^{-1}\mathbf{H}_{\pm}) d\xi_1 = -\frac{1}{12\mu}$$

Expression (5.13) takes the form $\frac{1}{12}(AH)'$. Eventually, by integrating (5.12), we complete the transformation of (5.9). Finally, the above-mentioned decay condition (5.9) for the boundary layer $\mathbf{V}^{-1}$ means that

$$\begin{aligned}
&\tfrac{1}{12}\partial_s[\mathrm{H}_1(y)^3\tau_{vs}(\nabla w; y)] + \sum_{i=v,s} \mathrm{H}_1(y)\partial_i\{-\tfrac{1}{2}(H_+(y) - H_-(y)) \times \\
&\times \tau_{vi}(\upsilon; y) + \tfrac{1}{4}(H_+(y)^2 + H_-(y)^2)\tau_{vi}(\nabla w; y)\} + \\
&+ \mathrm{H}_2(y)\partial_i\tau_{vi}(\upsilon; y) - \tfrac{1}{2}\mathrm{H}_3(y)\partial_i\tau_{vi}(\nabla w; y) = 0, \quad y \in \partial K \setminus 0
\end{aligned} \tag{5.14}$$

Thus, relationships (5.5)–(5.7) and (5.14) constitute the necessary system of boundary conditions for (2.12). They are, however, written in an inconvenient form (this becomes quite obvious if one recalls that in Sec. 2 it was necessary to reduce the system to self-adjoint form). We transform (5.5)–(5.7) and (5.14) to obtain natural boundary conditions for system (2.12). We remark that, by the condition for the energy E mentioned in Sec. 3 (it vanishes only for polynomials), this way of writing the problem proves that it is elliptic (cf. [12, Sec. 5.5]). We leave (5.5)–(5.7) unchanged and supplement (5.14) with Eqs (2.10) multiplied by $\frac{1}{2}(H_+ - H_-)$ and the result of applying the operator $\frac{1}{2}\partial_s(H_+ - H_-)$ to (5.7). Along with (5.4), we have

$$\partial_v(-\mathrm{H}_2\tau_{vv}(\upsilon) + \mathrm{H}_3\tau_{vv}(\nabla w)) + 2\partial_s(-\mathrm{H}_2\tau_{vs}(\upsilon) + \mathrm{H}_3\tau_{vs}(\nabla w)) = 0 \tag{5.15}$$

If N denotes the matrix-valued operator of the boundary conditions (5.5), (5.7), (5.15), and (5.6), then the vector-valued functions $(\upsilon^i, w^i) \in C_0^{\infty}(\overline{K} \setminus 0)^3$ will satisfy Green's formula

$$\langle T(\upsilon^1, w^1), (\upsilon^2, w^2)\rangle_K - \langle N(\upsilon^1, w^1), (\upsilon^2, w^2\partial_v w^2)\rangle_K = 2E(\upsilon^1, w^1; \upsilon^2, w^2)$$

We introduce the operator pencil $\Lambda \mapsto \| \mathbf{T}(\varphi, \partial_\varphi, \Lambda), \mathbf{N}_{\pm}(\varphi, \partial_\varphi, \Lambda) \|$ corresponding to problem (2.12), (5.5), (5.7), (5.15), in which the operator $\mathbf{N}_{\pm}$ is defined as in (3.4). Namely, we rewrite the boundary conditions under consideration in polar coordinates

$$r^{2m-1}\mathbf{N}_{\pm}^{j1}(\varphi, \partial_\varphi, r\partial_r)(\upsilon_r, \upsilon_\varphi) + r^{4m-2}\mathbf{N}_{\pm}^{j2}(\varphi, \partial_\varphi, r\partial_r)w = 0, \quad j = 1, 2$$

$$r^{4m-2}\mathbf{N}_{\pm}^{31}(\varphi,\partial_\varphi,r\partial_r)(\upsilon_r,\upsilon_\varphi)+r^{6m-3}\mathbf{N}_{\pm}^{32}(\varphi,\partial_\varphi,r\partial_r)w=0,$$

$$r^{4m-1}\mathbf{N}_{\pm}^{41}(\varphi,\partial_\varphi,r\partial_r)(\upsilon_r,\upsilon_\varphi)+r^{6m-2}\mathbf{N}_{\pm}^{42}(\varphi,\partial_\varphi,r\partial_r)w=0$$

Then the matrix-valued operator $\mathbf{N}_{\pm}$ consists of two blocks with components $\mathbf{N}_{\pm}^{k1}(\varphi,\ \partial_\varphi,\ \Lambda+1)$ and $\mathbf{N}_{\pm}^{k2}(\varphi,\ \partial_\varphi,\ \Lambda+2-2m)$. The operator pencil introduced retains all the properties of $\mathbf{T}$ mentioned in Sec. 3. Moreover, the polynomials (3.2) are power solutions of the homogeneous problem (2.12), (5.5)–(5.7), (5.15), and the asymptotic solutions $\mathbf{U}^1, \ldots, \mathbf{U}^6$ corresponding to them admit of the same physical interpretation as in Sec. 3.

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